Lecture 27

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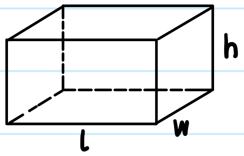
6.2 Volumes

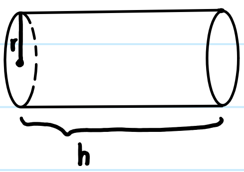
From your basic geometry, you probably know the volume for a rectangular prism



$V = lwh$

$= \text{Area of base} \times h$

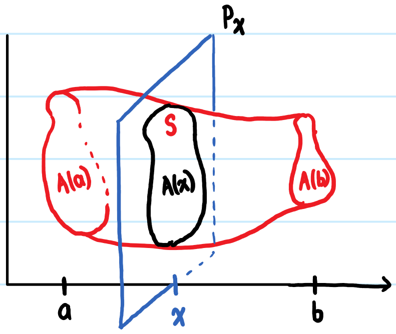
Similarly for cylinder,



$V = \text{Area of base} \times h$

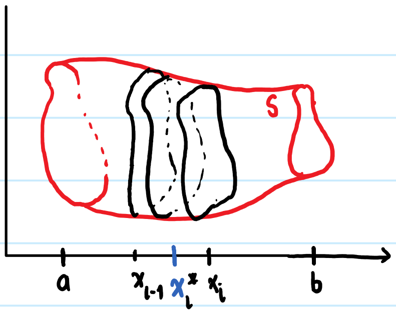
$= \pi r^2 h$

Idea For a solid S that is not a cylinder, we first "cut" S into pieces an approximate each piece by a cylinder.



Start by intersecting the solid w/ a plane obtaining a plane region, called <u>cross-section of S</u>. (Slicing a thin slice of cheese)

Let $A(x)$ denote the area of the cross section S in a plane P_x perpendicular to the x-axis and passing through the point x



1) Partition $[a,b]$ into n sub-intervals of equal width Δx. Pick a sample point x_i^* in the interval $[x_{i-1}, x_i]$

2) Now use the planes $P_{x_1}, \dots, P_{x_n}$ to divide S into n slabs. (This one is like slicing a loaf of bread)

3) Then we can approximate the i^{th} slab S_i by a cylinder with base area $A(x_i^*)$ and height Δx.

4) $V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$

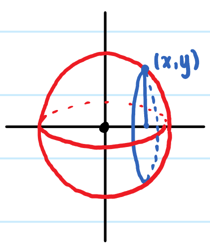
and the volume approximation becomes better and better as n becomes larger.

<u>DEFN</u> Let S be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, then the volume V of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

provided $A(x)$ is integrable.

<u>Ex</u> Find the volume of a sphere of radius r (centered at the origin).



1) Pick a point x and then the cross-section of sphere, perpendicular to the x-axis is a circle of radius y.

2) $x^2+y^2=r^2 \Rightarrow y=\sqrt{r^2-x^2}$

$A(x) = \pi y^2 = \pi(r^2-x^2)$

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3) Note that the sphere lies between $x = -r$ and $x = r$.
There,

$$V = \int_{-r}^{r} \pi (r^2 - x^2)\,dx = \pi \int_{-r}^{r} (r^2 - x^2)\,dx$$

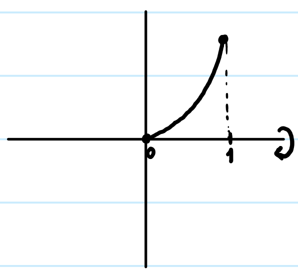
$$= 2\pi \int_{0}^{r} r^2 - x^2\,dx$$ [Since integrand is even and integral is symmetric]

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r = 2\pi \left[r^3 - \frac{r^3}{3} \right] = \frac{4\pi r^3}{3}$$

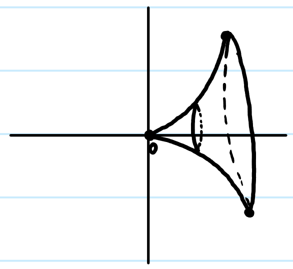
Volumes of solids of revolution:

What is a solid of revolution?

Ex Start w/ the curve $y = x^2$ from 0 to 1.



Rotate the curve about the x-axis.
Doing this for a curve gives us a solid.
We want to determine the volume of this solid.



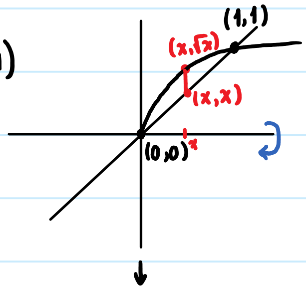
We need to slice through a point x, and when we do, we see that the cross section is a disc of radius $y = x^2$.

Then $A(x) = \pi y^2 = \pi x^4$ and

$$V = \int_0^1 \pi x^4\,dx = \pi \frac{x^5}{5} \Big]_0^1 = \frac{\pi}{5}$$

Ex Let $\mathcal{R}$ be the region bounded by the curves $y=\sqrt{x}$ and $y=x$.
Find the volume of the solid obtained by rotating the region about a) the x-axis b) y-axis

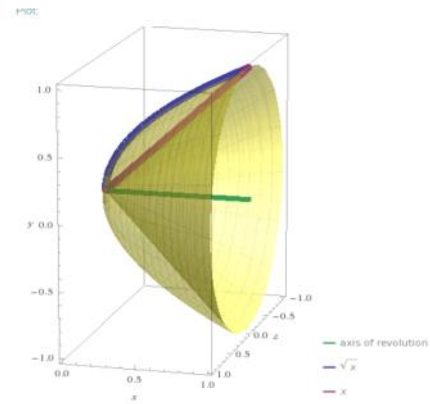
a)



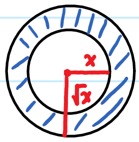
The curves $y=x$ and $y=\sqrt{x}$ intersect when

$$x=\sqrt{x} \Rightarrow x^2=x \Rightarrow x^2-x=0 \Rightarrow x(x-1)=0 \Rightarrow x=0, x=1.$$

So points of intersection are $(0,0)$ and $(1,1)$.



A cross-section has the shape of a washer (an annulus) with inner radius x and outer radius $\sqrt{x}$



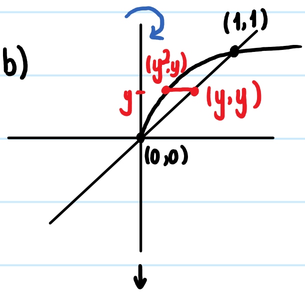
Then $A(x) = \pi(\sqrt{x})^2 - \pi(x)^2$

$= \pi x - \pi x^2 = \pi(x-x^2)$

Then, $V = \int_0^1 \pi(x-x^2)\,dx = \pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \pi\left[\frac{1}{2}-\frac{1}{3}\right]$

$= \frac{\pi}{6}$

b)



- As the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis and therefore to integrate with respect to y.

In this case, we slice at height y.
Again we get a washer with inner radius y^2 and outer radius y.
Then $A(y) = \pi(y)^2 - \pi(y^2)^2 = \pi(y^2-y^4)$
Since the solid lies betn $y=0,1$

$$V = \int_0^1 \pi(y^2-y^4)\,dy = \pi\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \frac{2\pi}{15}$$

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<u>Summary</u> : We calculate the volume of a solid of revolution by using the formula

$$V = \int_a^b A(x)\,dx \quad \text{or} \quad V = \int_c^d A(y)\,dy$$

and $A(x)$ or $A(y)$ is one of the following :

1) If cross-section is a disc, express radius in terms of x or y and $A = \pi(\text{radius})^2$

2) If cross section is a washer, express radii in terms of x or y.
   Let r_{out} be the outer radius and r_{in} be the inner radius.
   Then
   $$A = \pi r_{out}^2 - \pi r_{in}^2 = \pi\left[r_{out}^2 - r_{in}^2\right]$$

<u>Remark</u> : Slice the solid perpendicular to the axis of rotation.